Math 4650 - Homework # 4
Limits of functions

Part 1 - Computations

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1. For each $f : D \to \mathbb{R}$, a, L, and ϵ given, find a specific $\delta > 0$ so that if $x \in D$ and $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$.

   Draw a picture incorporating everything.

   (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$, $a = 2$, $L = 2$, $\epsilon = 0.01$.

   (b) $f : (0, \infty) \to \mathbb{R}$ given by $f(x) = 1/x$, $a = 1$, $L = 1$, $\epsilon = 0.1$.

Part 2 - Proofs

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2. Prove the following limit exists using the $\epsilon - \delta$ definition of limit.

   (a) $\lim_{x \to -1} (2x + 5) = 3$

   (b) $\lim_{x \to 1} \frac{5x}{x + 3} = \frac{5}{4}$

   (c) $\lim_{x \to 2} x^4$

   (d) $\lim_{x \to 1} \frac{1}{x^2}$

   (e) $\lim_{x \to 2} (x^3 - 1)$

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3. (Limits are unique)

   Prove that if $f : D \to \mathbb{R}$ and a is a limit point of D with $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} f(x) = L_2$, then $L_1 = L_2$.

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4. Let $f : D \to \mathbb{R}$ be a function. Suppose that $\lim_{x \to a} f(x) = L$ where $L \neq 0$. Prove that there exists $\delta > 0$ where if $x \in D$ and $0 < |x - a| < \delta$, then $|f(x)| > 0$.

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5. (A function with a limit at a must be bounded near a)

   Let $f : D \to \mathbb{R}$ be a function and a be a limit point of D. Suppose that $\lim_{x \to a} f(x)$ exists. Prove that there exists $M > 0$ and $\delta > 0$ such that if $x \in D$ and $0 < |x - a| < \delta$, then $|f(x)| < M$.

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6. (a) Let $D \subseteq \mathbb{R}$. Let $a \in \mathbb{R}$. Prove that a is a limit point of D if and only if there exists a sequence (x_n) contained in D with $x_n \neq a$ for all n and $x_n \to a$.

   (b) Prove that 1 is a limit point of $D = (1, 3]$.

   (c) Prove that 2 is not a limit point of $D = (-1, 1) \cup \{2\}$.
